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MODELS OF INVESTMENT UNDER UNCERTAINTY WHEN SHOCKS ARE NON-GAUSSIAN

INTRODUCTION

The investment literature of the last two decades has recognized the importance of interactions among the irreversibility of investment, uncertainty in the economic environment, and the choice of timing and/or scale of the new investment (see Pindyck (1982, 1988); Abel (1983); Bertola (1990); Dixit (1993); Dixit and Pindyck (1994); Abel and Eberly (1994); Bertola and Caballero (1994); Metcalf and Hassett (1995); Abel et al. (1996); Caballero and Pindyck (1996); see also the bibliography in Dixit and Pindyck (1994)).

In all of these papers, assumptions about the nature of the stochastic processes describing the economic environment are crucial. In the majority of the papers, the continuous time stochastic processes are used to model returns or prices. Usually, the geometric Brownian Motion models the movement of variables like the general price level, prices of financial instruments and option prices (see, e. g., Fisher (1975), Black and Scholes (1973); Merton (1973), Duffie (1992) and the bibliography there); for a discrete time analog, see Chow (1994). In some cases, the geometric Mean Reverting process is used — see, e. g., Dixit and Pindyck (1994); Metcalf and Hassett (1995).

The assumption that the exogeneous variable(s) of interest follow a Brownian Motion is very convenient since it allows one to obtain closed form solutions. At the same time, there is some empirical evidence against the modeling of observables as normal random variables.

As early as in 60th, Mandelbrot (1963) and Fama (1965) used stable Lévy processes to describe a slow decay of probability distribution densities of prices in financial markets (the so-called "fat tails"). The usage of these processes has two drawbacks, both due to the fact that the second moment of a Lévy stable distribution is infinite (unless it is a gaussian one):

1) the central limit theorem does not apply while it is well-documented that over longer time scales, financial data tend to become more Gaussian, and

2) it is impossible to construct a geometric process starting from a process with an infinite variance.

In 1994, Mantegna and Stanley (1994) constructed truncated Lévy processes. A truncated Lévy process has a finite variance but converges to a gaussian process very slowly. For the central part of the distribution its dynamics correspond to the dynamics predicted for a Lévy stable process, i. e. distributions have "fat tails" but the remote parts of the tails are cut off. Mantegna and Stanley (1995) find that the probability distribution of the Standard & Poor's 500 index can be described by a truncated Lévy process. The shorter the observed interval, the larger the deviation of financial-returns data from the benchmark normal distribution, exhibiting leptokurtosis (fat-tails). More specifically, in the central part of the distribution, the fall-off is governed by a power law (a normal distribution decays much faster, as $\exp(-\alpha x^2)$), and a Lévy stable distribution fits well. Eventually, in the tails of the distribution, the fall-off deviates from that characteristic of a Lévy stable process: it is approximately exponential, ensuring that (as one would expect for a price difference distribution) the variance of the distribution is finite. Later, Cont et al. (1997) developed a formula for the probability distribution of the Standard & Poor's 500 index futures, which explicitly describes the exponential fall-off and fits the data. They use another version of truncated Lévy processes suggested by Koponen (1995).

The existence of fat tails can be explained by the theory of "self-organized criticality" — see Bak (1996) and the bibliography therein — which deals with situations when power–law distributions appear. This happens in time series when very large fluctuations cannot, in general, be ignored in favor of the cumulative effect of the smaller ones. The emergence of power laws in economics can be explained within the framework of various economic models (e. g., see Romer (1931)). Bak suggests that in complex systems, catastrophes — that is, very large fluctuations — occur more frequently than implied by the Gaussian distribution.

However, in economic processes one should expect truncated fat tails because of the influence of barriers of various kind: extreme fluctuations are less probable than in physical models since new agents (e. g., firms) appear or some of the old ones disappear (exit) from the market, thus damping the fluctuations.

Thus, we have nice tractable models based on the assumption that observables are normal random variables, and a number of situations when observed processes exhibit fat tails.

In the paper we develop two discrete-time models which are almost as tractable as the most popular continuous time models based on the geometric Brownian Motion, yet they allow one to treat truncated Lévy flights and more general distributions. We do not make very specific assumptions on the probability distribution. In particular, we do not assume that it is possible to pass to the limit $\Delta t \to 0$ and describe the process by a continuous time model. We believe that this model is especially useful in cases when the time interval between observations is not very small, as it is the case in the theory of real options.

The first model is an approximate one and simpler; it is based on the smooth pasting condition assumption whereas in the framework of the second (rigorous) model we show that the smooth pasting condition fails in a discrete time model. For a discussion of the smooth pasting condition in the continuous-time gaussian model, see Dixit and Pindyck (1994); we are grateful to Avinash Dixit for pointing out that this condition may not hold in a discrete-time model.

Both models produce similar results.

In the first part of the paper, we develop a rigorous model and apply it to the Planner's Problem (see, e. g., Dixit and Pindyck (1994), Chapter 11), in the case of irreversible investment with zero operational cost, and show that the investment threshold may be very sensitive to a choice of the model describing the shocks.

We derive an explicit formula for the investment threshold, in terms of the observed distribution density. We produce numerical results for symmetric distributions from a three-parameter Koponen's (1995) family, which includes gaussian and truncated Lévy distributions. The results show that if a distribution is close to a Lévy distribution, i. e. the truncation happens far from the origin, the threshold increases — in some cases by dozens or even hundreds percent — as we replace a gaussian distribution with the truncated Lévy distribution, of the same variance. This means that it is optimal for investors to wait more than implied by the standard gaussian model (1994). If the truncation happens relatively close to the origin, then the threshold may decrease — not significantly, though.

The method is straightforward: we notice that the Bellman equation in the discrete-time model is the Wiener–Hopf equation, and can be solved by the Wiener–Hopf factorization method (1931) (which we use in a bit more modern form, Eskin (1973)), assuming that the investment threshold is given. The value function must satisfy certain conditions which lead to an equation for the threshold.

The same approach can be applied to the pricing of the perpetual American put option, under the same very weak assumptions on the probability distribution (see Levendorskii and Boyarchenko (1998)). There also exists a continuous-time version of the method; the corresponding results will be published separately.

In the second part, we consider the more realistic situations when

(i) a variable cost is present, and

(ii) the investment is partially reversible since of firms have exit as well as entry timing options. In other words, we examine not only the upper boundary (for the price of the output) which triggers new investment, but also the lower boundary which forces disinvestment or exit.

For the case of gaussian processes, this problem was considered by Dixit (1993) and Dixit and Pindyck (1994) (see also the bibliography therein); we modify the treatment of Dixit (1993) and derive a system of two nonlinear equations for the upper (entry) and lower (exit) thresholds.

Due to the high nonlinearity of the system when the higher and lower thresholds are unknowns, it is impossible to find an analytical solution to the system, and so, in applications one has to solve it numerically. This applies to both the standard model of Dixit (1993) and Dixit and Pindyck (1994) and to our model. We produce numerical examples which show that

a) the value of the upper (entry) threshold can increase significantly if we replace a gaussian with a non-gaussian distribution (with the same variance), hence optimal investment would be delayed, other things equal;

b) when investment is partially reversible with non-zero operational costs, the investment (entry) threshold always increases whereas the lower (exit) one may either increase or decrease, i. e., exit may either be furthered or delayed depending on the properties of the distribution;

c) whereas both the upper and lower Marshallian thresholds go to 0 as the volatility grows (implying that the regions where exit and inaction are optimal shrink, while entry becomes optimal for almost all parameter values), our model gives relatively stable thresholds. When the volatility becomes quite high, even the disinvestment threshold increases and exit occurs sooner. At high levels of volatility, the incentives to invest, and especially, to disinvest, become almost insensitive to further increases in volatility;

d) for the special case of completely irreversible investment with zero operational cost, we prove that at the high volatility limit, the investment threshold stabilizes at a finite value, and derive an approximate Tobin-type q formula for the upper threshold.

1. THE PLANNER'S PROBLEM: IRREVERSIBLE INVESTMENT

The model is a discrete version of the capacity choice model of investment described in Dixit and Pindyck (1994), Chapter 11. Consider a planner who chooses investment. The investment is irreversible, and each unit of capital costs K to install. The single period return when Q units of capital are in place is $XU(Q)$ where X is a stochastic shift variable. There is no operational cost, and a discount rate $r > 0$ is fixed. The planner's objective is to maximize the expected present value of returns net of capital installation cost.

Let $x_t = \ln X_t$, and assume that

$$x_{t+1} = x_t + \alpha + y_t, \tag{1}$$

where y_t are independent identically distributed random variables with zero mean and probability distribution density p satisfying

$$\int_{-\infty}^{+\infty} p(x)e^x \, dx < +\infty. \tag{2}$$

For simplicity, we consider symmetric p, though our results allow generalization to the case of non-symmetric distributions.

We have also to require

$$q := e^{-r+\alpha} \int_{-\infty}^{+\infty} p(x)e^x \, dx < 1. \tag{3}$$

The reason is that, for Q constant, the expected returns grow each period by a factor $e^{\alpha} \int_{-\infty}^{+\infty} p(x)e^x dx$, and are discounted back at rate e^{-r}, and hence are given by

$$w(x) = U(Q)e^x \sum_{j=0}^{+\infty} q^j;$$

for this series to converge, we need (3).

The last condition for p is: there exists $\varepsilon > 0$ such that

$$\int_{-\infty}^{+\infty} \left|p'(x)\right| e^{\varepsilon x}\, dx < +\infty; \tag{4}$$

this condition can be relaxed.

Let $w(Q, x)$ be the Bellman function. Due to the absence of variable cost,

(5) $\qquad w(Q, x)$ is non-decreasing w.r.t. Q, for x fixed,

and clearly,

(6) $\qquad w(Q, x)$ is non-decreasing w.r.t. x, for Q fixed,

and

(7) $\qquad w(Q, x)$ is non-negative.

On the strength of (6)–(7),

(8) $\qquad w(Q, x)$ is measurable and locally integrable w.r.t. x.

(If $w(Q, x) = +\infty$ for $x \geq b$, then for $c > b$ and $a < c$, the integral over (a, c) is $+\infty$.) Finally, assume that

(9) $\qquad U$ is differentiable and concave.

The argument on pp. 360–361 in Dixit and Pindyck (1994) shows that (9) imply

(10) $\qquad w(Q, x)$ is concave w.r.t. Q, for x fixed.

In discrete time, the Bellman equation for the problem under consideration is

$$\begin{aligned} w(Q, x_{j-1}) \;&=\; \max_{Q' \geq Q} \Big\{ e^{x_{j-1}} U(Q) \\ &\qquad -K \cdot (Q' - Q) + e^{-r} \mathbf{E} \left[w(Q', x_j) \mid x_{j-1} \right] \Big\}, \end{aligned} \tag{11}$$

where E is the expectation operator. Suppose that a point (Q, x) is in the inaction region, i. e. the maximum in (11) is attained at $Q' = Q$. Then

$$w(Q, x) = U(Q)\, e^x + e^{-r} \mathbf{E} \left[w(Q', x_j) \mid x_{j-1} \right],$$

or, on the strength of (1),

$$\begin{aligned} w(Q, x) \;&=\; U(Q) e^x + e^{-r} \int_{-\infty}^{+\infty} p(y) w(Q, x + \alpha + y)\, dy \\ &=\; U(Q) e^x + e^{-r} \int_{-\infty}^{+\infty} p(x + \alpha - y) w(Q, y)\, dy, \end{aligned} \tag{12}$$

for all $x < h(Q)$, where $x = h(Q)$ is the boundary of the inaction region. Thus, $H(Q) = e^{h(Q)}$ is the investment threshold, and to find it, we have to solve Eq. (12). To do this, we have to ensure that the value function is defined for all values of the capital and shocks.

Lemma 1.1. *Let (1), (2) and (9) hold, and let there exist (Q, x) such that $w(Q, x) < +\infty$.*

Then $w(Q, x) < +\infty$ for all (Q, x).

Proof. Suppose that for some y, $w(Q, y) = +\infty$. Then, on the strength of (6), $w(Q, z) = +\infty$, $\forall z > y$, and the RHS in (12) is infinite. The contradiction shows that $w(Q, x) < +\infty$, $\forall x$.

Due to (5), for $Q_1 < Q$, $w(Q_1, x) < +\infty$, and we may assume that Q_1 is in the action region. If (Q, x) and (Q_1, x) are in the action region,

$$\begin{aligned} w(Q, x) - w(Q_1, x) = K \cdot (Q - Q_1) + \big(U(Q) - U(Q_1)\big) e^x, \\ \forall x > h(Q). \end{aligned} \tag{13}$$

By dividing (13) by $Q - Q_1$ and passing to the limit $Q_1 \to Q$, we see that in the action region $w_Q(Q, x)$ exists and

$$w_Q(Q, x) = K + U'(Q)e^x, \qquad x > h(Q). \tag{14}$$

It follows from (9), (10) and (14), that there exists $C = C(Q_1, x)$ such that for all $Q > Q_1$, $w_Q(Q, x) < C$. By integrating, we obtain $w(Q, x) < +\infty$.

Lemma has been proved.

The disinvestment is never optimal since there is no variable cost and the installation cost cannot be recovered should X fall very low. Hence, h is non-decreasing, and for almost all Q, the derivative

$$h'(Q) \text{ exists.} \tag{15}$$

Below, we consider only Q satisfying (15) and derive a formula for $h(Q)$. We will see that the expression obtained defines a continuous function, therefore the formula will be valid for all Q.

Due to (5), for a given x, $w_Q(Q, x)$ exists for almost all Q, and by (14), for (Q, x) in the action region; it is possible to show that if Q satisfies (15), $w_Q(Q, x)$ exists for almost all $x < h(Q)$ (details are available on request). We choose Q satisfying (15), and differentiate (12) w.r.t. Q:

$$\begin{aligned} w_Q(Q, x) = U'(Q)e^x + e^{-r} \int_{-\infty}^{+\infty} p(x + \alpha - y) w_Q(Q, y)\, dy, \\ \forall x < h(Q). \end{aligned} \tag{16}$$

Due to (2), $\widehat{p} = \mathcal{F}p$, the Fourier transform of p:

$$\widehat{p}(k + i\tau) = \int_{-\infty}^{+\infty} e^{-ixk} p(x) e^{\tau x}\, dx,$$

is well-defined for all $k \in \mathbf{R}$ and $|\tau| \le 1$. Denote by $\mathcal{F}^{-1}$ the inverse Fourier transform, and for a function a, define an operator $a(D)$ by

$$\begin{aligned} a(D)u(x) &= \mathcal{F}^{-1} a(k) \mathcal{F} u(x) \\ &= (2\pi)^{-1} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{i(x-y)k} a(k) u(y)\, dy\, dk. \end{aligned}$$

An operator $a(D)$ is called a pseudo-differential operator with the symbol $a(D)$. If a is a polynomial, $a(D)$ is a differential operator. In particular, $Du = -iu'$.

By using the Taylor expansion, we obtain

$$u(x+\alpha) = \sum_{j=0}^{+\infty} \frac{1}{j!} u^{(j)}(x)\alpha^j = \big(\exp(i\alpha D)u\big)(x)$$

$(u^{(j)}$ denotes the derivative of order $j)$. Using this equality and an equality

$$\int_{-\infty}^{+\infty} p(x-y)u(y)\,dy = \big(\mathcal{F}^{-1}\widehat{p}(k)\mathcal{F}u\big)(x),$$

we may rewrite (16) as

$$\big(A(D)w_Q\big)(Q,x) = U'(Q)e^x, \qquad x < h(Q), \tag{17}$$

where $A(k) = 1 - e^{-r+i\alpha k}\widehat{p}(k)$.

If (17) were an equation on $\mathbf{R}$, we would have been able to solve it by applying an operator

$$A(D)^{-1} = \mathcal{F}^{-1}A(k)^{-1}\mathcal{F},$$

i. e. by making the Fourier transform, next multiplying by the inverse $A(k)^{-1}$, and then making the inverse Fourier transform. Unfortunately, the equality in (17) holds on an interval, and therefore, is more difficult to solve.

Fix Q and h, a prospective kandidat for $h(Q)$, and set

$$\begin{aligned} U' &= U'(Q), \\ u(x) &= w_Q(Q, x+h) - K - U'e^{x+h}. \end{aligned}$$

Since $De^{\beta x} = -i(e^{\beta x})' = -i\beta e^{\beta x}$, we obtain

$$\begin{aligned} A(D)K &= A(0)K, \\ A(D)e^x &= A(-i)e^x. \end{aligned}$$

Therefore, in terms of u, (14) and (17) are

(18) $$u(x) = 0, \qquad x > 0,$$

and

(19) $$\big(A(D)u\big)(x) = qU'e^{x+h} - r_1K, \qquad x < 0,$$

where $q = 1 - A(-i) = e^{-r+\alpha}\widehat{p}(-i)$ is the same as in (3), and $r_1 = A(0) = 1 - e^{-r}$.

Take small $\varepsilon \in (0,1)$, and set

$$\begin{aligned} u^\varepsilon(x) &= e^{\varepsilon x}u(x), \\ A^\varepsilon(k) &= A(k + i\varepsilon). \end{aligned}$$

By multiplying (18)–(19) by $e^{\varepsilon x}$ and taking into account that

$$\begin{aligned} e^{\varepsilon x}De^{-\varepsilon x}u(x) &= e^{\varepsilon x}\Big(-i\frac{d}{dx}\Big)e^{-\varepsilon x}u(x) \\ &= \Big(-i\frac{d}{dx} + i\varepsilon\Big)u(x) = (D + i\varepsilon)u(x), \end{aligned}$$

and hence, $e^{\varepsilon x}A(D)e^{-\varepsilon x} = A^\varepsilon(D)$, we obtain

(20) $$\big(A^\varepsilon(D)u^\varepsilon\big)(x) = qU'e^he^{(1+\varepsilon)x} - r_1Ke^{\varepsilon x}, \qquad x < 0,$$

(21) $$u^\varepsilon(x) = 0, \qquad x > 0.$$

To solve (20)–(21), we need the following lemma. It is a variant of standard factorization theorems (see, e. g., Eskin (1973), Section 6).

Lemma 1.2. *Let* (2) *and* (4) *hold.*

Then there exists $\varepsilon_0 > 0$ such that for any $|\varepsilon| \le \varepsilon_0$, $A^\varepsilon(k)$ admits a factorization

(22) $$A^\varepsilon(k) = A^\varepsilon_+(k)A^\varepsilon_-(k)$$

with the $A^\varepsilon_\pm(k)$ satisfying the following conditions:

a) *$A^\varepsilon_+(k + i\tau)$ (resp. $A^\varepsilon_-(k + i\tau)$) is holomorphic in a half-plane $\tau > 0$ (resp. $\tau < 0$), and admits a continuous extension into the closed half-plane;*

b) *there exist* $c > 0,$, C *such that*

$$c \le \left| A^{\varepsilon}_{\pm}(k \pm i\tau) \right| \le C, \qquad \forall \tau \ge 0; \tag{23}$$

c) $A^{\varepsilon}_{\pm}(k \pm i\tau)^{-1}$ *admits a representation*

$$A^{\varepsilon}_{\pm}(k \pm i\tau)^{-1} = 1 + T^{\varepsilon}_{\pm}(k \pm i\tau), \tag{24}$$

where $T^{\varepsilon}_{\pm}(k + i\tau)$ *is holomorphic in a half-plane* $\pm\tau > 0$, *and satisfies an estimate*

$$\left| T^{\varepsilon}_{\pm}(k + i\tau) \right| \le C\big(1 + |k| + |\tau|\big)^{-\omega_1}, \qquad \forall \pm \tau \ge 0, \tag{25}$$

where C *and* $\omega_1 > 0$ *are independent of* $k + i\tau$.

Proof. We set, for $\tau > 0$ and $k \in \mathbf{R}$,

$$\begin{aligned} b^{\varepsilon}_{\pm}(k \pm i\tau) \, &= \, \pm\frac{i}{2\pi}\int_{-\infty}^{+\infty} \frac{\ln A^{\varepsilon}(l)}{k \pm i\tau - l}\, dl, \\ A^{\varepsilon}_{\pm}(k \pm i\tau) \, &= \, \exp\big(b^{\varepsilon}_{\pm}(k \pm i\tau)\big). \end{aligned} \tag{26}$$

The proof that $A^{\varepsilon}_{\pm}$ satisfy (22) and a)–c) is a minor variation of the proof in Eskin (1973), Section 6 (details are available on request).

Parts a) and b) of Lemma 1.2 allow one to obtain a unique solution $u^{\varepsilon} \in L_2(R_-)$ to a problem (20)–(21) (see, e. g., Eskin (1973), Theorem 7.1):

$$u^{\varepsilon} = A^{\varepsilon}_{+}(D)^{-1}\theta_- A^{\varepsilon}_{-}(D)^{-1}\big(qU'e^{h}e^{(1+\varepsilon)x} - r_1 K e^{\varepsilon x}\big),$$

where θ_- is the characteristic function of a half-axis $\mathbf{R}_-$: $\theta_-(x) = 1$, $x < 0$, $\theta_-(x) = 0$, $x \ge 0$. By multiplying by $e^{-\varepsilon x}$, we obtain

$$u = e^{-\varepsilon x} A^{\varepsilon}_{+}(D)^{-1}\theta_- A^{\varepsilon}_{-}(D)^{-1}\big(qU'e^{h}e^{(1+\varepsilon)x} - r_1 K e^{\varepsilon x}\big). \tag{27}$$

Eq. (27) can be rewritten as

$$u = A^{0}_{+}(D)^{-1}\theta_-\left(A^{0}_{-}(-i)^{-1}qU'e^{h}e^{x} - A^{0}_{-}(0) r_1 K\right)$$

(details are available on request), so that u is independent of ε.

The derivation of a formula for $h = h(Q)$ is based on an analysis of the behaviour of $u = u(x, h)$ in the vicinity of zero. Since we have an explicit formula for the Fourier image of u, the analysis is straightforward but rather technical.

Lemma 1.3. *For $x > 0$, $u(x, h) = 0$, and as $x \to -0$,*

$$\begin{aligned} u(x,h) \,&=\, d(h)\big(1 + e^h\mu_1(x) + \mu_2(x)\big) \\ &\quad + xA_-^0(0)^{-1}r_1K + e^h\chi_1(x) + \chi_2(x), \end{aligned} \tag{28}$$

where functions $\mu_1(x) = o(1)$, $\mu_2(x) = o(1)$, $\chi_1(x) = o(x)$, $\chi_1(x) = o(x)$ are independent of h, and $d(h) = A_-^0(-i)^{-1}qU'e^h - A_-^0(0)^{-1}r_1K$.

Proof is available on request.

By returning to $w_Q(Q, x)$, we obtain

$$\begin{aligned} w_Q(Q,x,h) \,&=\, K + U'(Q)e^x + d(h)\big(1 + e^h\mu_1(x-h) + \mu_2(x-h)\big) \\ &\quad + (x-h)d(h)A_-^0(-i)^{-1}qU'(Q)e^h \\ &\quad + e^h\chi_1(x-h) + \chi_2(x-h), \end{aligned} \tag{29}$$

as $x \to h - 0$. Direct calculations (available on request) show that $b_-^0(-i)$ and $b_-^0(0)$ are real, therefore $A_-^0(-i)$ and $A_-^0(0)$ are positive, and $d(h)$ is real. Eq. (29) implies

$$\lim_{x\to h-0} w_Q(Q,x,h) \,=\, K + U'(Q)e^h + d(h),$$

and since

$$\lim_{x\to h+0} w_Q(Q,x,h) \,=\, K + U'(Q)e^h,$$

the assumption $d(h) \neq 0$ contradicts (10).

If $d(h) = 0$, (29) gives

$$\begin{aligned} \lim_{x\to h-0} w_Q(Q,x,h) \,&=\, K + U'(Q)e^h \\ &=\, \lim_{x\to h+0} w_Q(Q,x,h), \end{aligned}$$

and

$$\lim_{x\to h-0} w_{Qx}(Q,x,h) = A^0_-(0)^{-1}r_1K + U'(Q)e^h$$
$$> U'(Q)e^h = \lim_{x\to h+0} w_{Qx}(Q,x,h),$$

which agrees with (10) but shows that the smooth pasting condition (valid for a gaussian continuous-time model) fails in our discrete-time model.

Clearly, $d(h) = 0$ if and only if

$$H(Q)\big(=e^{h(Q)}\big) = \frac{r_1 A^0_-(-i)}{A^0_-(0)qU'}K, \tag{30}$$

and direct calculations (available on request) show that

$$\frac{A^0_-(-i)}{A^0_-(0)} = r_1^{-1/2}\exp(I_1 - I_2), \tag{31}$$

where

$$I_1 = \frac{1}{2\pi}\int_0^{+\infty} \ln\left(\left(1-e^{-r}\widehat{p}(l)\cos(\alpha l)\right)^2 + \left(e^{-r}\widehat{p}(l)\sin(\alpha l)\right)^2\right)$$
$$\times(1+l^2)^{-1}\,dl,$$
$$I_2 = \frac{1}{\pi}\int_0^{+\infty} \mathrm{arctg}\left(\frac{\widehat{p}(l)\sin(\alpha l)}{e^r - \widehat{p}(l)\sin(\alpha l)}\right)l^{-1}(1+l^2)^{-1}\,dl.$$

Theorem 1.1. *Let* (1)–(4) *hold.*
Then the investment threshold is given by (30)–(31).

Proof. We have proven (30) for almost all Q. Since h is non-decreasing and the RHS in (30) is continuous, (30) holds for all Q.

To facilitate the comparison with the Marshallian prescription, we rewrite (30) as

$$\frac{H(Q)U'(Q)}{1-q} = \frac{A^0_-(-i)r_1}{A^0_-(0)q(1-q)}K. \tag{32}$$

$XU'(Q)$ is the marginal utility, and the expected present value of it is equal to $XU'(Q)/(1-q)$ — see discussion after (3). A textbook Marshallian calculation tells the planner to invest when this value exceeds the cost K, but, as in Dixit (1993) and Dixit and Pindyck (1994), an additional factor

$$\varkappa = \frac{A^0_-(-i)r_1}{A^0_-(0)q(1-q)}$$

intervenes; in Dixit (1993) and Dixit and Pindyck (1994), the factor is $\varkappa_0 = \beta/(\beta-1)$, where $\beta > 1$ is a positive root to the characteristic equation $k^2\sigma^2/2 + \alpha k^2 - r = 0$.

Similar results were obtained in the first version of the present paper (Boyarchenko and Levendorskiĭ (1997)), where intuitively more appealing, albeit approximate, method was used. A formula for the factor was

$$\varkappa_1 = \frac{\beta_+}{\beta_+ - 1},$$

where $\beta_+ > 1$ was a positive root to the characteristic equation

$$\ln \widehat{p}(-i\mu) + \alpha\mu - r = 0.$$

It was shown that for a non-gaussian process with the same variance, $\beta_+ < \beta$, and hence, $\varkappa_1 > \varkappa_0$, and numerical results were produced to show that the difference may be dozens or even hundreds percent.

Here the factor, $\varkappa$, is rather complicated, and its comparative statics analysis would be difficalt to perform; still, it is not difficult to calculate it numerically.

Numerical Examples. The first truncated Lévy distributions were constructed by Mantegna and Stanley (1994). Later, Koponen (1995) constructed a family of truncated Lévy distributions which admitted an explicit description in terms of their Fourier transforms. For the sake of brevity, we consider only symmetric distributions of this family, with the Fourier transforms, $\widehat{p}_\nu$, given by

$$\widehat{p}_\nu(k) = \exp\left[-\sigma^2\lambda^2 \frac{1 - ((k/\lambda)^2+1)^{\nu/2}\cos(\nu \operatorname{arctg}(k/\lambda))}{\nu(\nu-1)}\right],$$

where $\sigma > 0$, $\lambda > 0$ and $\nu \in (0, 2]$, $\nu \neq 1$ are parameters. We have chosen a normalization so that the variance is independent of ν and λ.

For $\nu = 2$, we obtain

$$\widehat{p}_2(k) = \exp\left(-\frac{\sigma^2 k^2}{2}\right)$$

which means that p_2 is a gaussian distribution. As ν moves from 2 down, p_ν deviates from a gaussian distribution, and for fixed $\nu \in (0, 2)$, $\nu \neq 1$, in the limit $\lambda \to +0$, p_ν becomes a Lévy distribution with

$$\widehat{p}_\nu(k) = \exp\left(-\frac{c_1|k|^\nu \cos(\nu\pi/2)}{\nu(\nu-1)}\right).$$

Roughly speaking, $(-\lambda^{-1}, \lambda^{-1})$ is an interval where p_ν differs insignificantly from a Lévy distribution, and for $|x| \gg \lambda^{-1}$, the distribution exhibits an exponential fall-off.

Here are some numerical examples[1]. In the table below, we fix r, α, σ, λ and see how the factor $\varkappa$ varies with ν. Since Δt is normalized to unity, r, α, σ have to be small, which explains choices in examples below. For comparison, we give the corresponding values of the factor in Dixit (1993) and Dixit and Pindyck (1994), $\varkappa_0$; it depends on r, α and σ only.

Table 1. *Parameters*: $r = 0.006$, $\alpha = -0.002$.

ν	2.0	1.8	1.6	1.4	1.2	0.8	0.6	0.4	0.2	$\varkappa_0$
$\sigma = 0.095$ $\lambda = 1.5$	3.332	3.341	3.358	3.387	3.431	3.558	3.723	3.922	4.223	3.512
$\sigma = 0.111$ $\lambda = 1.5$	6.578	6.732	6.973	7.334	7.873	10.029	12.428	17.631	37.637	7.011
$\sigma = 0.079$ $\lambda = 2$	2.252	3.242	2.231	2.218	2.204	2.173	2.159	2.150	2.145	2.347

In the first two examples, it is clearly seen that the factor $\varkappa$ grows as ν goes from 2 down, i. e. as the process deviates from a gaussian one of the same variance.

[1] The authors thank Mitya Boyarchenko for help with the calculations.

In the first case, the factor $\varkappa$ can grow by more than 26%, and in the second case, by 572%.

The third example shows that the factor can decrease, though not that significantly — by 4.5%; as compared with the continuous time model — by 8.7%.

We see that the factor can increase quite dramatically as we replace a gaussian distribution with a non-gaussian one of the same variance. There are also cases when the factor decreases, though not significantly. This happens if a distribution is obtained from a Lévy one by truncation that is too close to origin.

Probably, the last effect is partly due to a smooth truncation in the Koponen's (1995) family: for a mixture $\nu p_2 + (1-\nu)pp$, $\nu \in [0,1]$, of a gaussian distribution p_2, with variance σ, and a distribution pp, uniform on $[-\sigma\sqrt{3}, \sigma\sqrt{3}]$, the threshold usually grows (weakly) as ν goes from 1 down and the distribution deviates from the gaussian one for various values of r, α, σ.

2. ENTRY AND EXIT STRATEGIES UNDER NON-GAUSSIAN DISTRIBUTIONS

Consider a planner who is contemplating investment (or disinvestment) of a unit of capital. Each unit of capital costs K to install, and the cost of abandoning a unit of capacity is J. If investment is partially reversible, J can be negative as there can be positive salvage value (usually, $-J < K$).

The present value of the cash flow when Q units of capital are installed is given by $XC(Q)$, where X is a stochastic shift variable. The interest rate $r(>0)$ is assumed to be constant. The variable cost per unit, v, is positive, so that installed capital may be reduced should X fall to very low levels.

The planner's objective is to maximize the present value of expected cash flows net of capital installation (or abandonment) costs, where valuation is by expected value under the equivalent martingale measure. When X rises, the planner must decide whether to install a unit of capital or not. In other words, she has to determine an upper threshold for X, $H(Q)$, which triggers new investment. Similarly, disinvestment will be triggered if a lower threshold, $L(Q)$, is hit.

We assume that X is observed at equally–spaced discrete time intervals t_j, with $t_j - t_{j-1} \equiv \Delta t$ being fixed (and small). We assume that $X_j \equiv X(t_j)$ satisfy

$$\ln X_j - \ln X_{j-1} = \alpha \Delta t + Y_j, \tag{33}$$

where Y_j are independent identically distributed (i. i. d.) random variables with zero mean and a probability density function $p(x)$ which admits a bound

$$p(x) \le D \exp(-\lambda |x|), \tag{34}$$

where D and $\lambda > 1$ are independent of x[2]. Note that α, p, D and λ depend on Δt, and that for gaussian distributions (34) holds with any λ.

Let $W(Q, X)$ denote the value of the project when at the initial state there is a demand shock at X and an amount Q of capacity is installed. Decomposing W into the cash flow over the next time interval, Δt, and the project continuation value, assuming that the cost is incurred at the beginning of each period, we have:

$$\begin{aligned} W(Q, X_{j-1}) \;=\; & \big(X_{j-1} C(Q) - vQ\big)\Delta t \\ & + \mathbf{E}\left[W(Q, X_j) \exp(-r\Delta t)\right], \end{aligned}$$

or

$$\begin{aligned} W(Q, X_{j-1}) \;=\; & \big(X_{j-1} C(Q) - vQ\big)\Delta t \\ & + \mathbf{E}\Big[W\big(Q, X_{j-1} \exp(\alpha \Delta t + Y_j)\big)\Big] \exp(-r\Delta t). \end{aligned}$$

If we fix j and denote $x \equiv \ln X_{j-1}$, we have

$$\begin{aligned} W(Q, \exp(x)) \;=\; & \big(\exp(x)\; C(Q) - vQ\big)\Delta t \\ & + \exp(-r\Delta t) \int_{-\infty}^{+\infty} p(y)\; W\big(Q, \exp(x + \alpha\Delta t + y)\big)\, dy. \end{aligned} \tag{35}$$

For given Q, (35) is a linear equation w. r. t. the unknown function $w(x) \equiv W(Q, \exp(x))$:

$$w(x) = \exp(-r\Delta t)\,(Pw)(x + \alpha\Delta t) + f(x), \tag{36}$$

[2] This condition is needed to ensure that the expectation values of X_{j+1} and $X_{j+1}^{\beta_\pm}$ below are finite.

where $f(x) \equiv (\exp(x)C(Q) - vQ)\Delta t$, and the linear operator $P \equiv P_{\Delta t}$ acts as follows:

$$(Pw)(x) \equiv \int_{-\infty}^{+\infty} p(y)w(x+y)\,dy.$$

Since (36) is a linear inhomogeneous equation, any solution is of the form:

$$w = w_1 + w_0$$

where w_1 is a particular solution to (36) (the same for all w) and w_0 is a solution to the corresponding homogeneous equation:

$$w(x) = \exp(-r\Delta t)\,(Pw)(x + \alpha\Delta t). \tag{37}$$

Due to the special form of $f(x) = (\exp(x)C(Q) - vQ)\Delta t$, it is natural to look for a particular solution w_1 of the form

$$w_1(x) = A\exp(x) + B, \tag{38}$$

where A and B are constants. Denote by $M(\mu)$ the moment generating function

$$M(\mu) = \int_{-\infty}^{+\infty} p(y)\exp(\mu y)\,dy,$$

and set $m(\mu) = \ln M(\mu)/\Delta t$[3].

By substituting (38) into (36) and using the equality

$$\int_{-\infty}^{+\infty} p(y)\exp\big(\mu(x+y)\big)\,dy = \exp\big(\mu x + m(\mu)\Delta t\big), \tag{39}$$

we obtain an expression for A:

$$A\Big(1 - \exp\big(\big(\alpha - r + m(1)\big)\Delta t\big)\Big)\exp(x) = \exp(x)\,C(Q)\Delta t,$$

giving

$$A = \frac{C(Q)}{c(r, \alpha, \Delta t, p, 1)},$$

[3] For a gaussian process, $m(\mu) = \frac{1}{2}\sigma^2\mu^2$.

where

$$c(r, \alpha, \Delta t, p, \mu) \equiv \frac{1 - \exp\left((\alpha\mu - r + m(\mu))\Delta t\right)}{\Delta t}.$$

Similarly, we find

$$B = \frac{-vQ}{c(r, \alpha, \Delta t, p, 0)}.$$

The particular solution is given by

$$w_1(x) = \frac{\exp(x)\, C(Q)}{c(r, \alpha, \Delta t, p, 1)} - \frac{vQ}{c(r, \alpha, \Delta t, p, 0)}. \tag{40}$$

(Note that due to Eq. (34), $m(1) < +\infty$.) This is the net present value of the project if no further investment or disinvestment is ever undertaken. For Q constant, the expected cash flow grows each period Δt by a factor $\exp((\alpha + m(1))\Delta t)$ and is discounted back at the rate $\exp(-r\Delta t)$, satisfying the equation

$$\begin{aligned} w(x) &= \exp(x)\, C(Q)\Delta t \\ &\quad + \exp\Big(\big(\alpha - r + m(1)\big)\Delta t\Big) w(x). \end{aligned}$$

The solution is the first term in the RHS of (40), while the second term is the discounted life-time cost. For the RHS in (40) to be positive and finite, we require

$$r - \alpha - m(1) > 0. \tag{41}$$

In realistic economic situations, $(r - \alpha - m(1))\Delta t$ and $r\Delta t$ are small, and therefore, approximately,

$$\begin{aligned} c(r, \alpha, \Delta t, p, 0) &= r, \\ c(r, \alpha, \Delta t, p, 0) &= r - \alpha - m(1), \end{aligned}$$

up to relatively small errors. If the process is gaussian, $m(1) = \sigma^2/2$, and

$$r - \alpha - m(1) = \frac{r - \alpha - \sigma^2}{2}$$

coincides with the denominator in the corresponding formula in Dixit (1993) and Dixit and Pindyck (1994).

Consider now the homogeneous linear equation (37), seeking for solutions of the form $w(x) = \exp(\mu x)$. By substituting $w(x) = \exp(\mu x)$ into (37) and using (39), we see that this function is a solution to eq. (37) if and only if μ is a root of the characteristic equation

$$1 - \exp\Big((\alpha\mu - r + m(\mu))\,\Delta t \Big) = 0. \tag{42}$$

The following theorem describes the set of real roots of eq. (42)[4].

Theorem 2.1. a) *Eq.* (42) *has at most one positive root,* β_+, *and one negative root,* β_-, *which coincide with the real roots of the equation*

$$m(\mu) + \alpha\mu - r = 0. \tag{43}$$

b) *Eq.* (42) *has a root on* $(-\lambda, 0)$ (*on* $(0, \lambda)$) *if and only if*

$$\lim_{\mu\to-\lambda} \big(m(\mu) + \alpha\mu - r \big) > 0 \quad \Big(\textit{resp.,} \ \lim_{\mu\to\lambda} \big(m(\mu) + \alpha\mu - r \big) > 0 \Big). \tag{44}$$

Equation (43) plays the role of the characteristic equation in the theory of ordinary differential equations; in Dixit and Pindyck (1994) this equation is dubbed the fundamental quadratic, since in cases arising there its order is equal to two.

In Boyarchenko and Levendorskii (1997), we argued that in realistic situations the non-real roots of eq. (42) have large imaginary parts. Such roots give rise to wildly oscillating solutions to the homogeneous equation. Disregarding these solutions as having little economic sense and assuming that (44) holds, we arrive at the following general solution of eq. (36):

$$w(x) = w_1(x) + A\exp(\beta_- x) + B\exp(\beta_+ x), \tag{45}$$

where w_1 is given by (40), and $\beta_\pm$ are the real solutions to eq. (43). Recall that (45) is valid under condition (41), and that w_1 and A, B depend on Q.

Having found the general solution (45) to eq. (36), we can next proceed similarly to Dixit (1993) and Dixit and Pindyck (1994). Returning to the

[4] See Theorem 2.1 in Boyarchenko and Levendorskiĭ (1997).

initial variables $X = \exp(x)$ and $W = W(Q, X)$, we can rewrite (45) as

$$
\begin{aligned}
W(Q,X) &= \frac{XC(Q)}{c(1)} - \frac{vQ}{c(0)} \\
&\quad + A(Q)X^{\beta_-} + B(Q)X^{\beta_+},
\end{aligned} \tag{46}
$$

where $A(Q)$ and $B(Q)$ are arbitrary constants, and $c(s) \equiv c(r, \alpha, \Delta t, p, s)$.

Eq. (46) for the value of capacity Q was derived assuming no local change in Q, therefore it is valid only in the region between the disinvestment and investment thresholds $(L(Q) < X < H(Q))$. By using the envelope condition for the Bellman equation at the upper and lower boundaries of the inaction region $L(Q) < X < H(Q)$, and then using the smooth pasting condition at both boundaries, we derive the following conditions:

$$
\begin{aligned}
W_Q(X,Q) &= K + XC'(Q)\Delta t, \\
W_{QX}(X,Q) &= C'(Q)\Delta t
\end{aligned} \quad \text{at } X = H(Q), \tag{47}
$$

$$
\begin{aligned}
W_Q(X,Q) &= -J + XC'(Q)\Delta t, \\
W_{QX}(X,Q) &= C'(Q)\Delta t
\end{aligned} \quad \text{at } X = L(Q). \tag{48}
$$

As $\Delta t \to 0$, (47) and (48) become the usual conditions in the continuous-time model in Dixit (1993) and Dixit and Pindyck (1994), i. e.,

at the upper threshold $X = H(Q)$: $W_Q = K$ and $W_{QX} = 0$,

and

at the lower threshold $X = L(Q)$: $W_Q = -J$ and $W_{QX} = 0$.

The four equations in (47) and (48) determine the functions $A(Q), B(Q)$, and the thresholds $H(Q), L(Q)$. By using (46), we can write down these equations explicitly:

$$
\begin{aligned}
&\frac{H(Q)C'(Q)(1 - c(1)\Delta t)}{c(1)} - \frac{v}{c(0)} + A'(Q)H(Q)^{\beta_-} \\
&\qquad + B'(Q)H(Q)^{\beta_+} = K;
\end{aligned} \tag{49}
$$

$$
\begin{aligned}
&\frac{C'(Q)(1 - c(1)\Delta t)}{c(1)} + \beta_- A'(Q)H(Q)^{\beta_- - 1} \\
&\qquad + \beta_+ B'(Q)H(Q)^{\beta_+ - 1} = 0;
\end{aligned} \tag{50}
$$

$$
\frac{L(Q)C'(Q)(1 - c(1)\Delta t)}{c(1)} - \frac{v}{c(0)} + A'(Q)L(Q)^{\beta_-}
$$

$$+ B'(Q)L(Q)^{\beta_+} = -J; \tag{51}$$

$$\frac{C'(Q)(1-c(1)\Delta t)}{c(1)} + \beta_- A'(Q)L(Q)^{\beta_- -1}$$
$$+ \beta_+ B'(Q)L(Q)^{\beta_+ -1} = 0. \tag{52}$$

Note that these equations are very similar to eq. (3.28)–(3.31) in Dixit (1993): only the constant factors are different, and the role of α, β in Dixit (1993) is played here by β_-, β_+, respectively.

If we set $K^* = K + v/c(0)$, $K_* = -J + v/c(0)$ and use Kramer's rule, we can solve the system of equations (49)–(52) with $A'(Q)L(Q)^{\beta_-}$, $A'(Q)H(Q)^{\beta_-}$, $B'(Q)L(Q)^{\beta_+}$, $B'(Q)H(Q)^{\beta_+}$ as the unknowns, and obtain

$$\begin{aligned}
A'(Q)L(Q)^{\beta_-} &= \frac{K_*\beta_+ + (1-\beta_+)L(Q)C'(Q)(1-c(1)\Delta t)/c(1)}{\beta_+ - \beta_-},\\
A'(Q)H(Q)^{\beta_-} &= \frac{K^*\beta_+ + (1-\beta_+)H(Q)C'(Q)(1-c(1)\Delta t)/c(1)}{\beta_+ - \beta_-},\\
B'(Q)L(Q)^{\beta_+} &= -\frac{K_*\beta_- + (1-\beta_-)L(Q)C'(Q)(1-c(1)\Delta t)/c(1)}{\beta_+ - \beta_-},\\
B'(Q)H(Q)^{\beta_+} &= -\frac{K^*\beta_- + (1-\beta_-)H(Q)C'(Q)(1-c(1)\Delta t)/c(1)}{\beta_+ - \beta_-}.
\end{aligned}$$

Next we can eliminate $A'(Q)$ and $B'(Q)$:

$$\left(\frac{H(Q)}{L(Q)}\right)^{\beta_-} = \frac{K^*\beta_+ + (1-\beta_+)H(Q)C'(Q)(1-c(1)\Delta t)/c(1)}{K_*\beta_+ + (1-\beta_+)L(Q)C'(Q)(1-c(1)\Delta t)/c(1)}, \tag{53}$$

$$\left(\frac{H(Q)}{L(Q)}\right)^{\beta_+} = \frac{K^*\beta_- + (1-\beta_-)H(Q)C'(Q)(1-c(1)\Delta t)/c(1)}{K_*\beta_- + (1-\beta_-)L(Q)C'(Q)(1-c(1)\Delta t)/c(1)}. \tag{54}$$

If we set $\omega_\pm = (\beta_\pm - 1)(1-c(1)\Delta t)/(\beta_\pm c(1))$ and divide the numerator and the denominator in the RHS of eq. (53) (respectively, eq. (54)) by β_+ (respectively, by β_-), we obtain

$$\left(\frac{H(Q)}{L(Q)}\right)^{\beta_-} = \frac{K^* - \omega_+ H(Q)C'(Q)}{K_* - \omega_+ L(Q)C'(Q)}, \tag{55}$$

$$\left(\frac{H(Q)}{L(Q)}\right)^{\beta_+} = \frac{K^* - \omega_- H(Q)C'(Q)}{K_* - \omega_- L(Q)C'(Q)}. \tag{56}$$

The system (55)–(56) is highly nonlinear and so, since analytically, it cannot be solved one has to resort to numerical simulations. The only exception is a special case of irreversible investment and zero operational cost, when there is no lower (exit) threshold and it is possible to write down an analytic expression for the upper (entry) threshold:

$$H(Q)C'(Q) = \frac{c(1)\beta_+}{\beta_+ - 1}K. \tag{57}$$

We consider this special case in more detail, before we present numerical simulations results.

2.1. A TOBIN-TYPE FORMULA FOR THE INVESTMENT THRESHOLD

Let parameters of the model vary in such a way that $\beta_+ \to 1$. Then, using an approximate formula $c(1) \sim r - \alpha - m(1)$, and applying the Lagrange formula to $m(\beta_+) + \alpha\beta_+ - r$ $(= 0)$, we obtain the approximate equality:

$$m'(1)(\beta_+ - 1) + \alpha(\beta_+ - 1) - c(1) = 0.$$

Hence,

$$\frac{c(1)\beta_+}{(\beta_+ - 1)} = m'(1) + \alpha,$$

and expression (57) for the investment threshold in the case of irreversible investment can be written as

$$H(Q) = \frac{qK}{C'(Q)},$$

where $q = m'(1) + \alpha$.

Recall that $m(\mu) = \ln M(\mu)/\Delta t$, so that

$$m'(1) = \frac{M'(1)}{M(1)\Delta t}, \tag{58}$$

and

$$M_{\Delta t}(1) = 1 + \frac{\sigma^2_{\Delta t}}{2} + \frac{\text{kurtosis}}{24} + \cdots, \tag{59}$$

$$M'_{\Delta t}(1) = \sigma^2_{\Delta t} + \frac{\text{kurtosis}}{6} + \cdots. \tag{60}$$

Typically, for small Δt one expects $\sigma^2_{\Delta t}$ to be small and the tails in (59) and (60) to be small w.r.t. $\sigma^2_{\Delta t}$. If this is the case, we can simplify (58) and write

$$q \approx \alpha + \frac{\sigma^2_{\Delta t}}{\Delta t} + \frac{\text{kurtosis}}{6\Delta t}. \tag{61}$$

If the process is close to a Gaussian one in the sense that the kurtosis is small relative to the variance, we can simplify (61) further and write

$$q \approx \alpha + \frac{\sigma^2_{\Delta t}}{\Delta t}. \tag{62}$$

Note that for a Gaussian process with constant parameters α and σ, the condition $\beta_+ \to 1$ is equivalent to $\sigma^2/2 \to r - \alpha$, so that (62) can be written in either of the following two forms:

$$q \approx \alpha + \sigma^2 \quad \text{or} \quad q \approx r + \frac{\sigma^2}{2}.$$

If the tails of the distribution are fat and the kurtosis is relatively large, its contribution to eq. (61) cannot be neglected. That is, for fat-tailed distributions, (61) should fit better than (62).

2.2. A NUMERICAL ILLUSTRATION

In Boyarchenko and Levendorskii (1997) we showed that for distributions of the Koponen's family (see numerical examples in Section 1)

$$m_\nu(\mu) = \frac{\sigma^2\lambda^2}{2\nu(\nu-1)}\big[(1+\mu/\lambda)^\nu + (1-\mu/\lambda)^\nu - 2\big].$$

We also added two (families of) distributions, p_0 and p_1; in terms of $m_\nu(\mu) = \ln M_\nu(\mu)/dt$, where $M_\nu(\mu)$ is the moment-generating function, they are given by

$$\begin{aligned} m_0(\mu) &= -\frac{\sigma^2\lambda^2}{2}\ln(1-\mu^2/\lambda^2), \\ m_1(\mu) &= \frac{\sigma^2\lambda^2}{2}\big[(1+\mu/\lambda)\ln(1+\mu/\lambda) \\ &\qquad +(1-\mu/\lambda)\ln(1-\mu/\lambda)\big]. \end{aligned}$$

These distributions also have constant variance σ^2.

We showed that for all ν, non-real roots of the characteristic equation (11) either do not exist or have large imaginary parts, and that $\beta_\pm$ exist if and only if either $\nu = 0$ or $\nu = 1$ and $2\ln 2 \pm \alpha - r > 0$ or $\nu \in (0, 2]$, $\nu \neq 1$ and $m_\nu(\lambda) \pm \alpha\lambda - r > 0$.

We also proved the following theorem which characterizes the dependence of roots $\mu_\pm = \mu_{\sigma,\lambda,\nu,\pm}$ and factors $\varkappa \equiv \beta_+/(\beta_+ - 1)$ on σ, λ, ν, with the other parameters being fixed.

Theorem 2.2. *If $\nu_1 \le \nu_2$, $\lambda_1 \le \lambda_2$, $\sigma_1 \ge \sigma_2$, and one of the inequalities is strict, then*

$$\mu_{\sigma_2,\lambda_2,\nu_2,-} < \mu_{\sigma_1,\lambda_1,\nu_1,-} < 0 < \mu_{\sigma_1,\lambda_1,\nu_1,+} < \mu_{\sigma_2,\lambda_2,\nu_2,+}.$$

If, in addition, $\mu_{\sigma_1,\nu_1,\lambda_1,+} > 1$, then

$$\varkappa_{\sigma_1,\lambda_1,\nu_1} > \varkappa_{\sigma_2,\lambda_2,\nu_2} > 1. \tag{63}$$

Equation (63) says that if one applies the standard model of irreversible investment of Dixit and Pindyck (1994) when the underlying stochastic process is actually a truncated Lévy process with $\nu < 2$, one obtains a lower investment threshold than the one predicted by our model. That is, if the distribution is non-gaussian, the firm should wait more before it invests. The smaller ν and λ, i.e. the greater the deviation from normality, the larger the discrepancy.

Below we present values for the higher (upper) and lower thresholds for the non-gaussian process, $H = H_\nu$ and $L = L_\nu$, and for the Marshallian thresholds,

$$HM = HM_\nu = \frac{K^* c(1)}{C'(Q)(1 - c(1)\Delta t)},$$
$$LM = LM_\nu = \frac{K_* c(1)}{C'(Q)(1 - c(1)\Delta t)},$$

as ν goes from 2 down to 0 (i. e. as the deviation from normality increases). We use values for r, α and σ that are typical for the examples in Dixit and Pindyck (1994).

Table 2. Upper (Entry) and lower (Exit) Investment Thresholds under a Non-Gaussian Distribution, H and L, vs. the Marshallian thresholds, HM and LM, for different values of ν. *Parameters*: $r = 0.05$, $\alpha = -0.05$, $\lambda = 2$, $K^* = 100$, $K_* = 35$, $C'(Q) = 2$.

Panel A ($\sigma = 0.3$)

°	2.0	1.8	1.6	1.4	1.2	1.0	0.8	0.6	0.4	0.2	0.0
H	5.93	6.02	6.13	6.28	6.45	6.64	6.80	7.08	7.35	7.62	7.92
HM	2.67	2.66	2.65	2.63	2.60	2.58	2.54	2.50	2.46	2.41	2.35
L	0.41	0.41	0.41	0.42	0.42	0.42	0.43	0.43	0.43	0.44	0.44
LM	0.93	0.93	0.92	0.91	0.90	0.89	0.88	0.86	0.84	0.82	0.80

Panel B ($\sigma = 0.4$)

°	2.0	1.8	1.6	1.4	1.2	1.0	0.8	0.6	0.4	0.2	0.0
H	6.68	6.75	6.84	6.95	7.09	7.26	7.45	7.66	7.90	8.17	8.46
HM	1.01	0.97	0.94	0.90	0.86	0.81	0.75	0.67	0.59	0.50	0.40
L	0.28	0.28	0.28	0.28	0.29	0.29	0.29	0.29	0.30	0.30	0.31
LM	0.35	0.34	0.33	0.32	0.30	0.28	0.26	0.24	0.21	0.17	0.14

From Table 4.A we see that H and L increase, whereas the Marshallian equivalents HM and LM decrease, as ν declines. The lower threshold (L) does not change significantly with ν, whereas the upper one (H) can increase by up to 33% as ν varies from 2.0 to 0.0.

More interesting still is the relation of the non-gaussian thresholds to the Marshallian thresholds: H is higher than the upper Marshallian threshold HM, suggesting that investment should be further delayed, while L is (usually) lower than the Marshallian threshold LM (implying that disinvestment should be postponed as well). Overall, the range of inaction may increase with deviation from normality and with volatility. The higher the volatility (see Table 4.B when σ increases from 0.3 to 0.4), the larger the discrepancy between the upper thresholds (whereas the discrepancy between lower thresholds may shrink).

CONCLUSION

In the dynamic stochastic environment of real life, random variables of complex systems are frequently characterized by power law distributions involving fat tails. Usually, such fat tails are caused by (and serve as indicators of) processes of self-organization; in economics, such processes appear when agents (e. g., firms) in the market are too sensitive to the behavior of other firms in their reference groups. The existence of fat-tailed distributions in economics is well-documented for interest rates and stock indices. At the same time, such fat tails are often truncated (eventually exhibiting exponential fall-off) due to the existence of entry and exit thresholds as a result of equilibrating mechanisms in the marketplace.

In the paper, we have constructed two discrete-time models of investment under uncertainty, which are applicable in the case of non-gaussian distributions. These models admit a closed-form solution as does the standard continuous-time model based on the geometric Brownian Motion (systematically used by Dixit and Pindyck (1994) and other authors). Our models allow one to treat more complex processes and do not require that passing to the continuous time limit be possible. The last remark is essential in applications where the time increment is not very small. In addition, in cases when the underlying stochastic process is a mixture of continuous and jump processes, our models do not require that this mixture be separated, as standard models do.

Here is another characterization of our models: in standard models of irreversible investment under uncertainty, only information about mean and variance is used, whereas in our models — about moments of higher order as well.

According to standard models, volatility changes the threshold for the investment: the higher volatility of a commodity's price (and the standard measure of volatility is the variance), the higher price level is needed to trigger new investment. Our models show that in the case of fat-tailed distributions, the threshold depends on the higher moments, and in some cases can be much higher still: it can grow with the higher moments even if the variance remains the same. However, there are also cases when the threshold decreases, and this may happen when fat tails are truncated in a small vicinity of the origin. This implies that policy interventions should aim at dumping large fluctuations rather than at decreasing the average volatility, i. e. variance.

REFERENCES

Abel A. (1983). Optimal investment under uncertainty. *American Economic Review* **73**, 228–233.

Abel, A. B., Dixit, A. K., Eberly, J. C., and Pindyck, R. S. (1996). Options, the value of capital, and investment. *Quarterly Journal of Economics* **111**, 753–777.

Abel, A., and Eberly, J. C. (1994). A unified model of investment under uncertainty. *American Economic Review* **84**, 1369–1384.

Bak, P. (1996). *How Nature Works: The Science of Self-Organized Criticality*, Copernicus/Springer, Berlin.

Bertola, G. (1990). Irreversible investment. *Mimeo.* Princeton University, Princeton, N.J.

Bertola, G., and Caballero, R. J. (1994). Irreversibility and aggregate investment. *Review of Economic Studies* **61**, 223–246.

Black, F., and Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy* **3**, 637–659.

Boyarchenko, S. I., and Levendorskiĭ, S. Z. (1997). A model of investment under uncertainty: the case of non-gaussian distributions. *Mimeo.*

Boyarchenko, S. I., and Levendorskiĭ, S. Z. (1998). Entry and exit strategies under non-gaussian distributions. In: *Innovation, Infrastructure and Strategic Options*. Ed. by M. J. Brennan, and L. Trigeorgis. Oxford University Press, London.

Caballero, R. J., and Pindyck, R. S. (1996). Uncertainty, investment, and industry evolution. *International Economic Review* **37**, 641–662.

Chow, G. C. (1994). Dynamic economics. *Mimeo*. Princeton University, Princeton, N.J.

Cont, R., Potters, M., and Bouchaud, J.-P. (1997). Scaling in stock market data: stable laws and beyond. In: *Scale Invariance and Beyond*. Ed. by B. Dubrulle, F. Graner, and D. Sornette. Springer, Berlin.

Dixit, A. K. (1993). Irreversible investment and competition under uncertainty. In: *Capital, Investment and Development*. Ed. by Kaushik Basu, Mukul Majumdar, and Tapan Mitra. Basil Blackwell, Cambridge, MA.

Dixit, A. K., and Pindyck, R. S. (1994). *Investment under Uncertainty*. Princeton University Press, Princeton, NJ.

Duffie, D. (1992). *Dynamical Asset Pricing Theory*. Princeton University Press, Princeton, NJ.

Eskin, G. I. (1980). *Boundary Problems for Elliptic Pseudo-Differential Equations*. Nauka, Moscow, 1973 (transl. of Math. Monograph 52, Amer. Math. Soc., Providence, Rhode Island).

Fama, E. F. (1965). The behavior of stock market prices. *Journal of Business* **38**.

Fisher, S. (1975). The demand for index bonds. *Journal of Political Economy* **83**, 509–534.

Koponen, I. (1995). Analytic approach to the problem of convergence of truncated Lévy flights towards the Gaussian stochastic process. *Physical Review E* **52**, 1197–1199.

Levendorskiĭ, S. Z., and Boyarchenko, S. I. (1998). On rational pricing of derivative securities for a family of non-gaussian processes. *Preprint Arbeitsgruppe Komplexe Analysis und Partielle Differentialgleichungen* 98/7. Potsdam Universität, Potsdam.

Mandelbrot, B. (1963). The variation of certain speculative prices. *Journal of Business* **36**, 392–417.

Mantegna, R. N., and Stanley, H. E. (1994). Stochastic process with ultraslow convergence to a gaussian: the truncated Lévy flight. *Phys. Rev. Lett.* **73**, 2946–2949.

Mantegna, R. N., and Stanley, H. E. (1995). Scaling behaviour in the dynamics of an economic index. *Nature* **376**, 46–49.

Merton, R. C. (1973). The theory of rational option pricing. *Bell Journal of Economics* **4**, 141–183.

Metcalf, G. E., and Hassett, K. A. (1995). Investment under alternative return assumptions. Comparing random walks and mean reversion. *Journal of Economic Dynamics and Control* **19**, 1471–1488.

Pindyck, R. S. (1982). Adjustment costs, uncertainty, and the behaviour of the firm. *American Economic Review* **72**, 415–427.

Pindyck, R. S. (1988). Irreversible investment, capacity choice, and the value of the firm. *American Economic Review* **78**, 969–985.

Romer, D. (1993). Rational Asset-Price Movements Without News, *American Economic Review* **83**, 1112–1130.

Wiener, N., and Hopf, E. (1931). Ueber eine Klasse singulärer Integralgleichungen. *Sitzungzberichte Preussische Akademie, Math. Phys. K1*, 696–706.

ON THE IMPACT OF THE POLICY UNCERTAINTY ON INVESTMENT

1. INTRODUCTION

In the paper, we consider the impact of policy uncertainty on investment decisions. One could argue that if firms are hesitant to invest under ongoing uncertainty, then the government should intervene and create additional investment incentives. However, this appears not necessarily to be true. Moreover, governments can even enhance uncertainty through the prospect of shifts in policy. The latter may be particularly relevant for the countries in the process of economic transition.
Here we will consider only changes in tax/ subsidy policy, following the treatment in Dixit and Pindyck (1994). We derive a system of equations for investment thresholds and solve it numerically assuming that the price of output follows non-gaussian process, and we show that in the presence of an additional source of uncertainty (in this case, policy uncertainty), difference in the threshold levels between the gaussian and non-gaussian cases is much larger than without this additional source.
In other words, an additional source of uncertainty enhances the effect of "non-gaussianness".

2. STATEMENT OF THE PROBLEM AND THE SYSTEM OF EQUATIONS FOR THE INVESTMENT THRESHOLDS

We begin with a firm contemplating a discrete investment project with sunk cost I. The firm is risk neutral. Assume that starting from moment t_0, the project will produce one unit of output in each time period (t_i, t_{i+1}). Let the instantaneous price of output equal P_{1i} and a flow cost equal P_{2i}. Denote

$$P_i = P_{1i} - P_{2i}$$

the profit flow at

$$t = t_i.$$

Suppose that P_i satisfy

(1) $$\ln P_{i+1} = \ln P_i + \alpha \, \Delta t + Y_i,$$

where Y_j are i.i.d. with zero mean and the probability distribution function $p(x)$ which satisfies

(2) $$p(x) \leq C \exp\left(-\lambda |x|\right),$$

and $C, \lambda > 1$ are independent of x.

Consider, as the policy instrument, an investment tax credit at rate θ $(0 < \theta < 1)$, which is given exogenously. If the tax credit policy is in effect, the firm incurs the a sunk cost of investment $(1-\theta)\, I$. However, the government has the possibility to switch from investment tax crediting to no credit policy regime and vice versa.

Policy uncertainty is not likely to be described by a Brownian Motion, therefore, following Dixit and Pindyck (1994), we assume that the shifts in policy follow a Poisson process. Namely, if the tax credit is currently not available (we further denote this state by a subscript 0), then the probability that it will be introduced during the next time period,

$$\Delta t = t_{i+1} - t_i,$$

is

$$1 - e^{-\lambda_1 \Delta t}.$$

Similarly, if the tax credit is currently in effect (we indicate this state by a subscript 1), then the probability that it will be withdrawn is

$$1 - e^{-\lambda_1 \Delta t}.$$

Intuitively, one could suggest the following firm's investment strategy (see Dixit and Pindyck (1994)):

a) there exists a lower threshold $\overline{P}^0$ for the profit flow such that below this threshold the firm will never invest irrespectively of the policy regime;

b) there also exists an upper threshold $\overline{P}^1$ such that above this value, the firm will always invest under both police regimes;

c) if the current profit flow P_j belongs to the interval $(\overline{P}^0, \overline{P}^1)$, then the firm will invest if and only if the tax credit is in effect.

To determine the thresholds $\overline{P}^0, \overline{P}^1$, we proceed as in Dixit and Pindyck (1994). Consider the net payoff to the investment opportunity at time t_j as a function of the current profit flow P_j. Denote this payoff by

$$V_1 = V_1(P_j)$$

if the tax credit is in effect, and by

$$V_0 = V_0(P_j)$$

otherwise.
If the investment project is undertaken, then the net payoffs can be calculated as follows:

$$\begin{aligned} V_0(P_j) &= V(P_j) - I, \\ V_1(P_j) &= V(P_j) - (1-\theta)\, I, \end{aligned}$$

where

$$V(P_j) = \mathbf{E}\left[\sum_{i=0}^{+\infty} P_{j+i} e^{-ir\Delta t}\right] \Delta t;$$

the discount rate r is constant.
Decomposing $V(P_j)$ as usual into the flow of profits over the next time interval Δt and the continuation value, we obtain

$$V(P_j) = P_j \Delta t + e^{-r\Delta t} V\big(\mathbf{E}\,[P_{j+1}]\big).$$

Using (1) and (2), we obtain

$$\mathbf{E}\,[P_{j+1}] = P_j \int_{-\infty}^{\infty} p(y)\, e^{\alpha \Delta t + y}\, dy$$

$$= P_j e^{\alpha \Delta t} M(1)$$
$$= P_j e^{\alpha \Delta t + m(1)\Delta t},$$

where

$$M(\mu) = M(p, \mu)$$

is the moment generating function defined (for $|\mu| < \lambda$) as

$$M(\mu) = \int_{-\infty}^{\infty} p(y)\, e^{\mu y}\, dy,$$

and

$$m(\mu) = \frac{\ln M(\mu)}{\Delta t}.$$

Therefore,

$$V(P_j) = P_j \Delta t + e^{(\alpha - r + m(1))\Delta t} V(P_j),$$

provided

$$1 - e^{(m(1) + \alpha - r)\Delta t} > 0$$

or equivalently,

(3) $$m(1) + \alpha - r < 0.$$

(This condition ensures that the NPV is finite.)
If (3) holds, then

$$V(P_j) = P_j \, \frac{\Delta t}{1 - q},$$

where

$$q = e^{(m(1) + \alpha - r)\Delta t},$$

hence

(4) $$V_0(P_j) \; = \; P_j \, \frac{\Delta t}{1 - q} - I,$$

(5) $$V_1(P_j) \; = \; P_j \, \frac{\Delta t}{1 - q} - (1 - \theta)\, I.$$

Consider separately the cases when
1) $P_j > \overline{P}^1$,
2) $P_j \in (\overline{P}^0, \overline{P}^1)$,
and
3) $P_j \in (0, \overline{P}^0)$.
If

$$P_j > \overline{P}^1,$$

then the firm always invests, so $V_0(P_j), V_1(P_j)$ are given by equations (4) and (5), respectively.
If

$$P_j \in (\overline{P}^0, \overline{P}^1),$$

then the firm invests when tax credit is available, so $V_1(P_j)$ is given by (5) as above. However, $V_0(P_j)$ is more complicated now.
During the next short time interval Δt, with probability

$$1 - e^{-\lambda_1 \Delta t}$$

the credit will be implemented, the firm will invest and its value will be $V_1(P_{j+1})$. Otherwise, the firm will not invest and its value will be $V_0(P_{j+1})$. Thus

$$\begin{aligned} V_0(P_j) = e^{-r\Delta t}\Big\{ & (1 - e^{-\lambda_1 \Delta t})\, \mathbf{E}\left[V_1(P_{j+1})\right] \\ & + e^{-\lambda_1 \Delta t} \mathbf{E}\left[V_0(P_{j+1})\right]\Big\}. \end{aligned} \tag{6}$$

Shifting (5) one period forward and using (2), we obtain

$$\mathbf{E}\left[V_1(P_{j+1})\right] = e^{(m(1)+\alpha)\Delta t} P_j \frac{\Delta t}{1-q} - (1-\theta)\, I. \tag{7}$$

Fix j and denote

$$x = \ln P_j, \quad w_k(z) = V_k(e^z), \qquad k = 0, 1;$$

then

$$\begin{aligned} \mathbf{E}\left[V_k(P_{j+1})\right] &= \int_{-\infty}^{+\infty} p(y)\, w_k(x+\alpha\Delta t+y)\, dy \\ &= (\mathcal{P}w_k)\,(x+\alpha\Delta t), \end{aligned} \tag{8}$$

where a linear operator $\mathcal{P}$ acts as follows

$$(\mathcal{P}w)\,(x) = \int_{-\infty}^{+\infty} p(y)\, w(x+y)\, dy.$$

Substituting (7) and (8) into (6), we get the following equation with respect to an unknown function $w_0(x)$:

$$\begin{aligned} w_0(x) &= e^{(-r-\lambda_1)\Delta t}(\mathcal{P}w_0)\,(x+\alpha\Delta t) \\ &\quad +f_1(x)+f_2(x), \end{aligned} \tag{9}$$

where

$$\begin{aligned} f_1(x) &= (1-e^{-\lambda_1\Delta t})\, e^{(m(1)+\alpha-r)\Delta t}\,\frac{\Delta t}{1-q}, \\ f_2(x) &= (\theta-1)\, e^{-r\Delta t}(1-e^{-\lambda_1\Delta t})\, I. \end{aligned}$$

We look for a solution to (9) in the form of a linear combination of exponents, and we use an equality

$$\mathcal{P}\, e^{\mu(x+a)} = e^{\mu(x+a)+m(\mu)\Delta t}, \tag{10}$$

for $|\mu| < \lambda$.
The special forms of functions f_1, f_2 as well as the form of equation (9) suggest that it is natural to look for a solution in the form

$$w_0(x) = w_{00}(x) + w_{01}(x) + w_{02}(x), \tag{11}$$

where

$$\begin{aligned} w_{01}(x) &= c_1 e^x, \\ w_{02}(x) &= c_2, \end{aligned}$$

and $w_{00}(x)$ is a linear combination of exponents $A_\mu e^{\mu x}$ satisfying the homogeneous equation

$$w(x) = e^{-(r+\lambda_1)\Delta t}(\mathcal{P}w)\,(x+\alpha\Delta t). \tag{12}$$

Substituting (12) into (11) and equating the coefficients at the same exponents in the LHS and the RHS, we obtain c_1 and c_2 as

$$\begin{aligned} c_1 &= (1-e^{-\lambda_1\Delta t})\,e^{(m(1)+\alpha-r)\Delta t}\,\frac{\Delta t}{1-q}\,\frac{1}{1-e^{(m(1)+\alpha-r-\lambda_1)\Delta t}}, \\ c_2 &= e^{-r\Delta t}(1-e^{-\lambda_1\Delta t})\,\frac{\theta-1}{I(1-e^{-(r+\lambda_1)\Delta t})}; \end{aligned}$$

and exponents μ are roots of an equation

$$1-e^{(\alpha\mu-r-\lambda_1)\Delta t}M(\mu)=0.$$

the real roots of those satisfy

$$m(\mu)+\alpha\mu-r-\lambda_1=0. \tag{13}$$

As was shown in Boyarchenko and Levendorskii (1997), the number of positive roots is at most 1, and the same is true for negative roots. Assuming that both of these exist, we can write

$$V_0(P) = AP^{\beta_+}+BP^{\beta_-}+c_1P+c_2, \tag{14}$$

where $\beta_\pm$ are roots of Eq. (13).

Finally, we consider

$$P_j\in(0,\overline{P}^0).$$

Here the firm waits in both policy regimes, and regimes can switch both ways. Following the same steps as above, we derive

$$\begin{aligned} V_0(P_j) &= e^{-r\Delta t}\Big\{(1-e^{-\lambda_1\Delta t})\,\mathbf{E}\left[V_1(P_{j+1})\right] \\ &\qquad +e^{-\lambda_1\Delta t}\mathbf{E}\left[V_0(P_{j+1})\right]\Big\}, \end{aligned}$$

$$\begin{aligned} V_1(P_j) &= e^{-r\Delta t}\Big\{(1-e^{-\lambda_0\Delta t})\,\mathbf{E}\left[V_0(P_{j+1})\right] \\ &\qquad\qquad +e^{-\lambda_0\Delta t}\mathbf{E}\left[V_1(P_{j+1})\right]\Big\}. \end{aligned}$$

Fix j and set

$$x = \ln P_j, \quad w_k(z) = V_k(e^z).$$

We can rewrite the last two equations as follows:

$$\begin{aligned} w_0(x) &= e^{-r\Delta t}(1-e^{-\lambda_1\Delta t})\,(\mathcal{P}w_1)\,(x+\alpha\Delta t) \\ &\quad +e^{-(r+\lambda_1)\Delta t}(\mathcal{P}w_0)\,(x+\alpha\Delta t), \end{aligned}$$

$$\begin{aligned} w_1(x) &= e^{-r\Delta t}(1-e^{-\lambda_0\Delta t})\,(\mathcal{P}w_0)\,(x+\alpha\Delta t) \\ &\quad +e^{-(r+\lambda_0)\Delta t}(\mathcal{P}w_1)\,(x+\alpha\Delta t). \end{aligned}$$

This is a system of linear equations:

(15) $$w(x) = (\mathcal{P}Aw)\,(x+\alpha\Delta t),$$

where

$$w = \begin{pmatrix} w_0 \\ w_1 \end{pmatrix}, \quad \mathcal{P}w = \begin{pmatrix} \mathcal{P}w_0, \\ \mathcal{P}w_1 \end{pmatrix}$$

and

$$A = [a_{jk}]_{j,k=1,2}$$

is a constant matrix with the entries

$$\begin{aligned} a_{11} &= e^{(-r-\lambda_1)\Delta t}, \\ a_{22} &= e^{(-r-\lambda_0)\Delta t}, \\ a_{12} &= 1-a_{11}, \\ a_{21} &= 1-a_{22}. \end{aligned}$$

Direct calculations show that the eigenvalues of A are

$$\begin{aligned} s_1 &= 1, \\ s_2 &= -1+a_{11}+a_{22}, \end{aligned}$$

and the corresponding eigenvectors are

$$u^1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad u^2 = \begin{pmatrix} -\dfrac{1-a_{11}}{1-a_{22}} \\ 1 \end{pmatrix},$$

We look for a solution to system (15) in the form of a linear combination

$$w(x) = v_1(x)\, u^1 + v_2(x)\, u^2,$$

where v_j are scalar functions.
Substituting $w(x)$ into (15), we see that v_j are solutions to equations

$$v_1(x) \;=\; e^{-r\Delta t}(\mathcal{P}v_1)\,(x+\alpha\Delta t), \tag{16}$$

$$v_2(x) \;=\; e^{(-r-\overline{\lambda})\Delta t}(\mathcal{P}v_2)\,(x+\alpha\Delta t), \tag{17}$$

where

$$\overline{\lambda} = -\frac{\ln\left(a_{22}+a_{11}-1\right)}{\Delta t}.$$

As above, we look for solutions of Eq. (16) in the form of a linear combination of exponents $\exp(\mu x)$, with real μ. They are roots to the equation

$$m(\mu) + \alpha\mu - r = 0. \tag{18}$$

Since we look for solutions on an interval $(-\infty, \ln \overline{P}^0)$, and this solution has to be bounded, we must take positive roots only.
The positive root is unique, and we assume that it exists. Denoting it by β_1, we obtain

$$v_1(x) = Ce^{\beta_1 x},$$

where C is an arbitrary constant.
Similarly, the general solution to Eq. (17) is

$$v_1(x) = De^{\beta_2 x},$$

where D is an arbitrary constant, and β_2 is the positive root of the equation

$$m(\mu) + \alpha\mu - r - \overline{\lambda} = 0. \tag{19}$$

(Once again we assume that this root exists.)
By combining formulas for w, v_1 and v_2, we obtain the following formulas for V_0 and V_1 on an interval $(0, \overline{P}^0)$:

$$V_0(P) = CP^{\beta_1} - D\,\frac{1 - e^{-\lambda_1 \Delta t}}{1 - e^{-\lambda_0 \Delta t}}\,P^{\beta_2}, \tag{20}$$

$$V_1(P) = CP^{\beta_1} + D\,P^{\beta_2}. \tag{21}$$

By writing down the discrete version of "smooth pasting conditions" (see Boyarchenko and Levendorskiĭ (1997)) at

$$P = \overline{P}^0$$

and

$$P = \overline{P}^1,$$

we obtain a system of 6 equations with 6 unknowns $A, B, C, D, \overline{P}^0, \overline{P}^1$. The system is highly non-linear, so only numerical methods can be used to solve it. In the last section, we provide an example which shows that the effect of "non-gaussianness" can be seriously enhanced by an additional source of uncertainty — in this case, by the policy uncertainty.

3. A NUMERICAL EXAMPLE

Let $p_{\nu,\lambda,\sigma}$ be a family of non-gaussian distributions introduced by Koponen (1995) for

$$\nu \in (0, 2], \quad \nu \neq 1, \quad \lambda > 0 \quad \text{and} \quad \sigma > 0,$$

and supplemented for

$$\nu = 0,\, 1$$

in Boyarchenko and Levendorskiĭ (1997); the notation was introduced in Boyarchenko and Levendorskiĭ (1997).
As was shown in Boyarchenko and Levendorskiĭ (1997), the variance is equal to σ^2, and if

$$\nu = 2,$$

the distributions are gaussian ones. When ν goes from 2 down, the distribution deviates from a gaussian distribution, and "the non-gaussiannes" increases as λ decreases.

In the following example, we take values

$$r = 0.05, \quad \sigma = 0.2,$$

fairly typical for examples in Dixit and Pindyck (1994). We also take

$$\alpha = -m(1)$$

which means that the drift of P is zero (this is also typical for examples in Dixit and Pindyck (1994)), and $I = 20$, so that the Marshallian threshold is equal to $rI = 1$. Finally, we take

$$\begin{aligned} \theta &= 0.1, \\ \Delta t &= 1, \\ \lambda &= 6.5, \\ \lambda_0 &= 0.01, \\ \lambda_1 &= 0.2. \end{aligned}$$

In the following table, in the first row we let ν run from 2 till 0, in the second row, we give values of the investment threshold $\bar{P}$ for the case when there is no tax policy at all (these values are calculated on the basis of the formula in Boyarchenko and Levendorskii (1997)), and in the third and fourth lines, the values of $\overline{P}^0$ and $\overline{P}^1$ are given.

ν	2.0	1.6	1.4	0.8	0.4	0.0
$\overline{P}$	1.863	1.877	1.900	1.929	1.965	2.009
$\overline{P}^0$	1.670	1.683	1.703	1.730	1.763	1.810
$\overline{P}^1$	3.101	3.182	3.300	3.443	3.603	3.779

The table shows that

1) with no tax policy at all, the investment threshold can increase by 7.8% as the process deviates from a gaussian one, of the same variance;

2) when the tax credit is in effect, the investment threshold can increase by 8.4% as the process deviates from a gaussian one, of the same variance;

3) when the tax credit is not in effect, the investment threshold can increase by 21.7% as the process deviates from a gaussian one, of the same variance.

Note that for the case when tax credit is in effect, the uncertainty (=the probability of withdrawal) is very small, and still, the effect of "non-gaussiannes" has increased from 7.8% to 8.4%; when tax credit is not in effect and the uncertainty (= the probability that tax credit will be introduced) is larger, the effect of "non-gaussiannes" has become almost three times larger.
Similar effects can be observed for other parameter values.

REFERENCES

Boyarchenko, S. I., and Levendorskii, S. Z. (1997). A model of investment under uncertainty: the case of non-gaussian distributions. *Working paper.*

Dixit, A. K., and Pindyck, R. S. (1994). *Investment under Uncertainty.* Princeton University Press, Princeton, NJ.

Koponen, I. (1995). Analytic approach to the problem of convergence of truncated Lévy flights towards the Gaussian stochastic process. *Physical Review E* **52**, 1197–1199.